

QUASICROSSED PRODUCT ON G -GRADED QUASIALGEBRAS

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ABSTRACT. The notion of quasicrossed product is introduced in the setting of G -graded quasialgebras, i.e., algebras endowed with a grading by a group G , satisfying a “quasiasociative” law. The equivalence between quasicrossed products and quasicrossed systems is explored. It is presented the notion of graded-bimodules in order to study simple quasicrossed products. Deformed group algebras are stressed in particular.

1. INTRODUCTION

The G -graded quasialgebras were introduced by H. Albuquerque and S. Majid about a decade ago [4], and during the last years have been addressed with some collaborators (see [2] and the references therein). Inspired by the theory of graded rings and graded algebras [9, 10, 11, 12], in the present paper we extend the concepts of crossed products and crossed systems to the context of G -graded quasialgebras. The division graded quasialgebras are quasicrossed products, as well as some notable nonassociative algebras such as the deformed group algebras, for example, Cayley algebras and Clifford algebras. Among them, we stress the octonions with potential relevance to many interesting fields of mathematics, namely Clifford algebras and spinors, Bott periodicity, projective and Lorentzian geometry, Jordan algebras, and the exceptional Lie groups. We also refer to its applications to physics, such as, the foundations of quantum mechanics and string theory. There are some others interesting examples of quasicrossed products, such as the simple antiassociative superalgebras. We prove some basic results about quasicrossed products and quasicrossed systems, emphasizing the case of deformed group algebras. Our work extends the study on unital antiassociative quasialgebras with semisimple even part presented in [7].

In Section 2, we collect the basic definitions and properties related to graded quasialgebras. Section 3 is devoted to a review of some properties of the set of the units of a graded quasialgebra. In Section 4 the quasicrossed products and systems are defined and interrelated. Some examples of quasicrossed products are also included. Then, in Section 5, we establish a relation of equivalence between quasicrossed systems, a relation between quasicrossed products and, in the end, we relate them in a suitable way. Section 6 reveals some compatibility between quasicrossed products and the Cayley-Dickson process. It is shown that the quasicrossed system corresponding to the deformed group algebra obtained from the Cayley-Dickson process applied to a deformed group algebra is related to the quasicrossed system corresponding to the initial algebra. Section 7 is dedicated to simple quasicrossed products. The definition of representation of a graded quasialgebra is introduced and described in a commutative diagram. Some examples of graded modules over graded quasialgebras are included. Finally, in Section 8, central simplicity (as quasialgebras) for deformed group algebras corresponding to quasicrossed systems is explored.

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2. PRELIMINARIES

Throughout this work, A denotes an algebra with identity element 1 over an algebraically closed field \mathbb{K} with characteristic zero. We also fix a multiplicative group G with neutral element e .

Definition 2.1. A *grading* by a group G of an algebra A is a decomposition $A = \bigoplus_{g \in G} A_g$ as a direct sum of vector subspaces $\{A_g \neq 0 : g \in G\}$ of A indexed by the elements of G satisfying

$$A_g A_h \subset A_{gh}, \quad \text{for any } g, h \in G,$$

where we denote by $A_g A_h$ the set of all finite sums of products $x_g x_h$ with $x_g \in A_g$ and $x_h \in A_h$. An algebra A endowed with a grading by a group G is called a *G -graded algebra*. Moreover, if it satisfies the stronger condition

$$A_g A_h = A_{gh}, \quad \text{for any } g, h \in G,$$

we call A a *strongly G -graded algebra*.

In this paper G is generated by the set of all the elements $g \in G$ such that $A_g \neq 0$, usually called the *support* of the grading. The subspaces A_g are referred to as *homogeneous component* of the grading. The elements of $\bigcup_{g \in G} A_g$ are called the *homogeneous elements* of A . A nonzero $x_g \in A_g$ is *homogeneous* of degree g and is denoted by $\deg x_g = g$. Any nonzero element $x \in A$ can be written uniquely in the form $x = \sum_{g \in G} x_g$, where $x_g \in A_g$ and at most finitely many elements x_g are nonzero. Given two gradings Γ and Γ' on A , Γ is a *refinement* of Γ' if any homogeneous component of Γ' is a (direct) sum of homogeneous components of Γ . A grading is *fine* if it admits no proper refinement. Throughout this paper, the gradings will be considered fine.

A subspace $B \subseteq A$ is called a *graded subspace* if $B = \bigoplus_{g \in G} (B \cap A_g)$. Equivalently, a subspace B is graded if for any $x \in B$, we can write $x = \sum_{g \in G} x_g$, where x_g is a homogeneous element of degree g in B , for any $g \in G$. The usual regularity concepts will be understood in the graded sense. That is, a *graded subalgebra* is a subalgebra which is a graded subspace, and an *ideal* $I \subset A$ is a graded subspace $I = \bigoplus_{g \in G} I_g$ of A such that $IA + AI \subset I$.

Definition 2.2. [4] A map $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$ is called a *cocycle* if

$$\phi(h, k, l)\phi(g, hk, l)\phi(g, h, k) = \phi(g, h, kl)\phi(gh, k, l), \quad (2.1)$$

$$\phi(g, e, h) = 1, \quad (2.2)$$

hold for any $g, h, k, l \in G$, where e is the identity of G .

Next lemma lists some properties of cocycles useful in the sequel.

Lemma 2.3. *If $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$ is a cocycle then the following conditions hold for any $g, h \in G$:*

- (i) $\phi(e, g, h) = \phi(g, h, e) = 1$;
- (ii) $\phi(g, g^{-1}, g)\phi(g^{-1}, g, h) = \phi(g, g^{-1}, gh)$;
- (iii) $\phi(g, g^{-1}, g)\phi(g^{-1}, g, g^{-1}) = 1$;
- (iv) $\phi(h, h^{-1}, g^{-1})\phi(g, h, h^{-1}) = \phi(g, h, h^{-1}g^{-1})\phi(gh, h^{-1}, g^{-1})$.

Proof. First we show (i). In (2.1) we consider $k = e$ and get

$$\phi(h, e, l)\phi(g, h, l)\phi(g, h, e) = \phi(g, h, l)\phi(gh, e, l).$$

Now by (2.2) it comes $\phi(g, h, e) = 1$. We obtain the other equality in a similar way. To show (ii) we replace in (2.1) h by g^{-1} , k by g , l by h and take in account (i). The item (iii)

is a particular case of (ii) with $h = g^{-1}$. Now we prove (iv) from the definition of cocycle. For any $g, h \in G$

$$\begin{aligned}\phi(h, h^{-1}, g^{-1})\phi(g, h, h^{-1}) &= \phi(h, h^{-1}, g^{-1})\phi(g, hh^{-1}, g^{-1})\phi(g, h, h^{-1}) \\ &= \phi(g, h, h^{-1}g^{-1})\phi(gh, h^{-1}, g^{-1})\end{aligned}$$

□

The category of G -graded vector spaces is monoidal by way of

$$\begin{aligned}\Phi_{V,W,Z} : (V \otimes W) \otimes Z &\longrightarrow V \otimes (W \otimes Z) \\ (v_g \otimes w_h) \otimes z_k &\longmapsto \phi(g, h, k)v_g \otimes (w_h \otimes z_k),\end{aligned}$$

for any homogeneous elements v_g of degree g in V , w_h of degree h in W and z_k of degree k in Z .

Definition 2.4. A map $F : G \times G \longrightarrow \mathbb{K}^\times$ is a 2-cochain if

$$F(e, g) = F(g, e) = 1,$$

holds for any $g \in G$.

The notion of graded quasialgebra was introduced in [4]. This new concept includes the usual associative algebras but also some notable nonassociative examples, like the octonions.

Definition 2.5. Let $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$ be an invertible cocycle. A G -graded quasialgebra (or just a graded quasialgebra) is a G -graded algebra $A = \bigoplus_{g \in G} A_g$ with product map $A \otimes A \longrightarrow A$ obeying to the quasiassociative law in the sense

$$(x_g x_h) x_k = \phi(g, h, k) x_g (x_h x_k), \quad (2.3)$$

for any $x_g \in A_g, x_h \in A_h, x_k \in A_k$. Moreover, a graded quasialgebra A is called *coboundary* if the associated cocycle is

$$\phi(g, h, k) = \frac{F(g, h)F(gh, k)}{F(h, k)F(g, hk)},$$

for a certain 2-cochain F and $g, h, k \in G$.

Remark 2.6. If A is a G -graded quasialgebra then A_e is an associative algebra ($1 \in A_e$) and A_g is an associative A_e -bimodule for any $g \in A_g$.

Example 2.7. All associative graded algebras are graded quasialgebras (with $\phi(g, h, k) = 1$ for any $g, h, k \in G$). In particular for the group $G = \mathbb{Z}_2$, the graded quasialgebras admit only two types of algebras. The mentioned associative case with ϕ identically 1, and the antiassociative case with $\phi(x, y, z) = (-1)^{xyz}$, for all $x, y, z \in \mathbb{Z}_2$. The antiassociative quasialgebras were studied in [3] and recently addressed in [1]. For $G = \mathbb{Z}_3$, every cocycle has the form

$$\begin{aligned}\phi_{111} &= \alpha, \quad \phi_{112} = \beta, \quad \phi_{121} = \frac{1}{\omega\alpha}, \quad \phi_{122} = \frac{\omega}{\beta}, \\ \phi_{211} &= \frac{\alpha}{\beta\omega}, \quad \phi_{212} = \alpha\omega, \quad \phi_{221} = \frac{\beta}{\omega\alpha}, \quad \phi_{222} = \frac{\omega}{\alpha},\end{aligned}$$

for some nonzero $\alpha, \beta \in \mathbb{K}$ and ω a cubic root of the unity. Here ϕ_{111} is a shorthand for $\phi(1, 1, 1)$, etc. \mathbb{Z}_n -quasialgebras are studied in [5].

Lemma 2.8. A G -graded quasialgebra A is strongly graded if and only if $1 \in A_g A_{g^{-1}}$ for all $g \in G$.

Proof. Suppose $1 \in A_g A_{g^{-1}}$ holds for all $g \in G$. For any $h \in G$ it follows then that

$$A_{gh} = 1A_{gh} \subset A_g A_{g^{-1}} A_{gh} \subset A_g A_h,$$

hence $A_{gh} = A_g A_h$. The converse is obvious. □

Lemma 2.9. *Let A be a strongly graded and commutative quasialgebra, then G is an abelian group.*

Proof. Since A is strongly graded, we have that $A_g A_h = A_{gh} \neq 0$ for any $g, h \in G$. Therefore there exist $x_g \in A_g$ and $x_h \in A_h$ such that $x_g x_h \neq 0$. From A is commutative, we have that $x_g x_h = x_h x_g \neq 0$, and this implies $gh = hg$. \square

Lemma 2.10. *Let A be a strongly G -graded quasialgebra. If $x \in A$ is such that $x A_g = 0$ or $A_g x = 0$, for some $g \in G$, then $x = 0$.*

Proof. Let $x \in A$ such that $x A_g = 0$ for some $g \in G$ (the another case is analogue). We have then $x A_g A_{g^{-1}} = 0$, or equivalently $x A_e = 0$. From $1 \in A_e$, we conclude that $x = 0$. \square

Remark 2.11. By the above result we have for a strongly G -graded quasialgebra A that always is $A_g \neq 0$, for any $g \in G$, even g does not belong to the support of the grading.

3. UNITS OF A GRADED QUASIALGEBRA

Definition 3.1. An element u of a graded quasialgebra A is called a *left unit* if there exists a left inverse $u_L^{-1} \in A$ meaning $u_L^{-1} u = 1$. Similarly, u is said a *right unit* if there exists a right inverse $u_R^{-1} \in A$ meaning $u u_R^{-1} = 1$. By an *unit* (or *invertible element*) we mean an element $u \in A$ such that has a left and right inverses. We denote by $U(A)$ the set of all units of A .

Definition 3.2. We say that a unit u of A is *graded* if $u \in A_g$ for some $g \in G$. The set of all graded units of A is denoted by $Gr U(A)$ and we have $Gr U(A) = \bigcup_{g \in G} (U(A) \cap A_g)$.

Remark 3.3. We have that $U(A)$ is not, in general, a group because the product does not satisfy the associative law. The left and right inverses of a unit are not necessary equals.

Lemma 3.4. *Let u be a graded unit of degree g of a graded quasialgebra A . The following assertions hold.*

- (i) *The left and right inverses u_L^{-1} and u_R^{-1} of u have degree g^{-1} and are related by $u_R^{-1} = \phi(g^{-1}, g, g^{-1}) u_L^{-1}$.*
- (ii) *The left and right inverses u_L^{-1}, u_R^{-1} of u are unique.*
- (iii) *If w is another graded unit of A of degree h , then the product uw is a graded unit of degree gh such that,*

$$(uw)_L^{-1} = \frac{\phi(g^{-1}, g, h)}{\phi(h^{-1}, g^{-1}, gh)} w_L^{-1} u_L^{-1}, \quad (uw)_R^{-1} = \frac{\phi(h, h^{-1}, g^{-1})}{\phi(g, h, h^{-1} g^{-1})} w_R^{-1} u_R^{-1}.$$

- (iv) *The set $Gr U(A)$ is closed under products and inverses.*

Proof. (i) We show that $u_L^{-1} \in A_{g^{-1}}$ (it is similar for the right inverse). We may write $u_L^{-1} = \sum_{h \in G} u_h$, where $u_h \in A_h$ and at most finitely many elements u_h are nonzero. From $1 = u_L^{-1} u = \sum_{h \in G} u_h u$ it follows that $u_h = 0$ unless $h = g^{-1}$. Thus $u_L^{-1} = u_{g^{-1}}$ has degree g^{-1} . The quasiassociativity of A gives

$$u_R^{-1} = 1 u_R^{-1} = (u_L^{-1} u) u_R^{-1} = \phi(g^{-1}, g, g^{-1}) u_L^{-1} (u u_R^{-1}) = \phi(g^{-1}, g, g^{-1}) u_L^{-1}$$

as desired (cf. [6, 7]).

(ii) Suppose that exist u_L^{-1} and $u'_L{}^{-1}$ two left inverses of u , meaning that $u_L^{-1} u = 1$ and $u'_L{}^{-1} u = 1$. Then $u_L^{-1} u = u'_L{}^{-1} u$. Since u is a unit of A , exists u_R^{-1} satisfying $u u_R^{-1} = 1$. We may write $(u_L^{-1} u) u_R^{-1} = (u'_L{}^{-1} u) u_R^{-1}$, hence $\phi(g^{-1}, g, g^{-1}) u_L^{-1} (u u_R^{-1}) = \phi(g^{-1}, g, g^{-1}) u'_L{}^{-1} (u u_R^{-1})$ and we obtain $u_L^{-1} = u'_L{}^{-1}$. The case with the right unit is analogue.

(iii) As $A_g A_h \subset A_{gh}$ then uw is a homogeneous element of degree gh . Since A is quasiassociative, we get the expression of the left inverse of uw doing

$$\begin{aligned} (w_L^{-1} u_L^{-1})(uw) &= \phi(h^{-1}, g^{-1}, gh) w_L^{-1} (u_L^{-1}(uw)) = \frac{\phi(h^{-1}, g^{-1}, gh)}{\phi(g^{-1}, g, h)} w_L^{-1} ((u_L^{-1} u)w) \\ &= \frac{\phi(h^{-1}, g^{-1}, gh)}{\phi(g^{-1}, g, h)} w_L^{-1} w = \frac{\phi(h^{-1}, g^{-1}, gh)}{\phi(g^{-1}, g, h)}. \end{aligned}$$

In a similar way we obtain the right inverse of uw (cf. [6, 7]).

(iv) By (iii) we conclude that $Gr U(A)$ is closed under products. To show that $Gr U(A)$ is closed under inverses, meaning that whenever u is a graded unit then u_L^{-1} and u_R^{-1} are graded units too, we use (i) and observe that

$$\left(\phi(g^{-1}, g, g^{-1})u \right) u_L^{-1} = uu_R^{-1} = 1$$

and

$$u_R^{-1} \left(\frac{1}{\phi(g^{-1}, g, g^{-1})} u \right) = u_L^{-1} u = 1$$

completing the proof. \square

Remark 3.5. From Lemma 3.4(i)-(ii), the left and right inverses of any $u \in U(A) \cap A_g$ are also units of A and

$$\begin{aligned} (u_L^{-1})_R^{-1} &= u, & (u_L^{-1})_L^{-1} &= \phi(g^{-1}, g, g^{-1})u, \\ (u_R^{-1})_L^{-1} &= u, & (u_R^{-1})_R^{-1} &= \frac{1}{\phi(g^{-1}, g, g^{-1})}u. \end{aligned}$$

Lemma 3.6. *In the case A is a graded associative algebra, left and right inverses are equal.*

Proof. It is easy to check that $u_L^{-1} = u_L^{-1}(uu_R^{-1}) = (u_L^{-1}u)u_R^{-1} = u_R^{-1}$ for any $u \in U(A)$. \square

Corollary 3.7. *In the case $u \in U(A) \cap A_e$ then its left and right inverses are equal and belong to A_e . Moreover, $U(A) \cap A_e = U(A_e)$.*

Proof. By Lemma 3.4(i), the left and right inverses of u belong to A_e and $u_R^{-1} = \phi(e, e, e)u_L^{-1} = u_L^{-1}$. Therefore $U(A) \cap A_e \subseteq U(A_e)$. The reciprocal is trivial. \square

Remark 3.8. The map $deg : Gr U(A) \rightarrow G$ preserves the multiplication and the elements $u \in Gr U(A)$ such that $deg u = e$ are the set $U(A) \cap A_e = U(A_e)$.

Lemma 3.9. (i) *The map $\mu : Gr U(A) \rightarrow Aut(A_e)$ defined by*

$$\mu(u)(x) = x u u_R^{-1} \text{ for any } u \in Gr U(A) \text{ and } x \in A_e,$$

satisfies $\mu(uw) = \mu(u) \circ \mu(w)$ for all $u, w \in Gr U(A)$.

(ii) *Right multiplication by any $u \in A_g \cap U(A)$ is an isomorphism*

$$A_e \rightarrow A_e u = A_g$$

of left A_e -modules.

Proof. (i) Let u, w be two graded units of A such that $\deg u = g$ and $\deg w = h$. Using Lemma 2.3(iv) we obtain for any $x \in A_e$,

$$\begin{aligned}
\mu(uw)(x) &= (uw)x(uw)_R^{-1} = (uw)x \frac{\phi(h, h^{-1}, g^{-1})}{\phi(g, h, h^{-1}g^{-1})} (w_R^{-1}u_R^{-1}) \\
&= \frac{\phi(h, h^{-1}, g^{-1})}{\phi(g, h, h^{-1}g^{-1})} (uw) \left(x(w_R^{-1}u_R^{-1}) \right) \\
&= \frac{\phi(h, h^{-1}, g^{-1})}{\phi(g, h, h^{-1}g^{-1})} (uw) \left((xw_R^{-1})u_R^{-1} \right) \\
&= \frac{\phi(h, h^{-1}, g^{-1})}{\phi(g, h, h^{-1}g^{-1})\phi(gh, h^{-1}, g^{-1})} \left((uw)(xw_R^{-1}) \right) u_R^{-1} \\
&= \frac{\phi(h, h^{-1}, g^{-1})\phi(g, h, h^{-1})}{\phi(g, h, h^{-1}g^{-1})\phi(gh, h^{-1}, g^{-1})} \left(u(xw_R^{-1}) \right) u_R^{-1} \\
&= u(xw_R^{-1})u_R^{-1} = \mu(u) \circ \mu(w)(x)
\end{aligned}$$

(ii) First we prove that the right multiplication is a monomorphism. Let $x, y \in A_e$ such that $xu = yu$. Thus $(xu)u_R^{-1} = (yu)u_R^{-1}$. Since $x, y \in A_e$ then $x(uu_R^{-1}) = y(uu_R^{-1})$ and $x = y$. To prove that it is an epimorphism, we need to see if for any $v \in A_g$ exists $x \in A_e$ such that $xu = v$. We get it taking $x = vu_R^{-1}$. \square

4. QUASICROSSED PRODUCTS AND QUASICROSSED SYSTEMS

In this section we introduce the concept of quasicrossed product in the context of graded quasialgebras.

Definition 4.1. Let A be a G -graded quasialgebra. We say that A is a *quasicrossed product* of G over A_e if for any $g \in G$ exists $\bar{g} \in U(A) \cap A_g$, meaning that, there exists an unit \bar{g} in A of any degree g .

The following examples illustrate that some important quasialgebras are quasicrossed product.

Example 4.2. Any division graded quasialgebra $A = \bigoplus_{g \in G} A_g$ is trivially a quasicrossed product of G over A_e , because $1 \in A_e$ and every nonzero homogeneous element is invertible.

Example 4.3. Interesting examples of division graded quasialgebras, so of quasicrossed products, are deformed group algebras $\mathbb{K}_F G$ (see [4]). We present properly this class of algebras since we will pay special attention to them in this paper. Consider the group algebra $\mathbb{K}G$, the set of all linear combinations of elements $\sum_{g \in G} a_g g$, where $a_g \in \mathbb{K}$ such that $a_g = 0$ for all but finitely many elements g . We define $\mathbb{K}_F G$ with the same underlying vector space as $\mathbb{K}G$ but with a modified product $g.h := F(g, h)gh$, for any $g, h \in G$, where F is a 2-cochain on G . Then $\mathbb{K}_F G$ is a coboundary graded quasialgebra. Moreover, any $\mathbb{K}_F G$ is a quasicrossed product. In fact, given $g \in G$ and $a_g \in \mathbb{K}^\times$ then the homogeneous element $a_g g \in (\mathbb{K}_F G)_g$ is an unit with left inverse and right inverse:

$$(a_g g)_L^{-1} = \phi(g, g^{-1}, g)(a_g g)_R^{-1} = F(g^{-1}, g)^{-1} a_g^{-1} g^{-1}.$$

There are two classes of modified group algebras particularly interesting, namely the Cayley algebras and the Clifford algebras. We mention just some well studied Cayley algebras:

- (1) The complex algebra \mathbb{C} is a $\mathbb{K}_F G$ quasialgebra with $G = \mathbb{Z}_2$ and $F(x, y) = (-1)^{xy}$, for $x, y \in \mathbb{Z}_2$.
- (2) The quaternion algebra \mathbb{H} is a $\mathbb{K}_F G$ quasialgebra with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $F(\vec{x}, \vec{y}) = (-1)^{x_1 y_1 + (x_1 + x_2) y_2}$, where $\vec{x} = (x_1, x_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ is a vector notation.

- (3) The octonion algebra \mathbb{O} is another $\mathbb{K}_F G$ algebra for $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $F(\vec{x}, \vec{y}) = (-1)^{\sum_{i \leq j} x_i y_j + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3}$, where $\vec{x} = (x_1, x_2, x_3) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Any Clifford algebra is another quasialgebra of type $\mathbb{K}_F G$ for $G = (\mathbb{Z}_2)^n$ and 2-cochain $F(\vec{x}, \vec{y}) = (-1)^{\sum_{i \leq j} x_i y_j}$ where $\vec{x} = (x_1, \dots, x_n) \in (\mathbb{Z}_2)^n$. Recall that \mathbb{C} and \mathbb{H} are both Cayley and Clifford algebras.

Example 4.4. Let $\text{Mat}_n(\Delta)$ be the \mathbb{Z}_2 -graded algebra of the $n \times n$ matrices over Δ with the natural \mathbb{Z}_2 -gradation inherited from Δ , where $\Delta = \Delta_{\bar{0}} \oplus \Delta_{\bar{1}}$ is a division antiassociative quasialgebra ($\simeq \langle D, \sigma, a \rangle$ see [3]) and $n \in \mathbb{N}$. Consider $\text{Mat}_n(\Delta) = \text{Mat}_n(\Delta_{\bar{0}}) \oplus \text{Mat}_n(\Delta_{\bar{0}})u$ equipped with multiplication defined by

$$\begin{aligned} A(Bu) &= (AB)u \\ (Au)B &= (A\bar{B})u, \\ (Au)(Bu) &= aA\bar{B}, \end{aligned} \quad \text{for all } A, B \in \text{Mat}_n(\Delta_{\bar{0}}),$$

where the matrix \bar{B} is obtained from the matrix $B = [b_{ij}]_{1 \leq i, j \leq n}$ by replacing the term b_{ij} by $\sigma(b_{ij})$, for all $i, j \in \{1, \dots, n\}$. Then the simple antiassociative quasialgebra $\text{Mat}_n(\Delta)$ is clearly a quasicrossed product of \mathbb{Z}_2 over $\text{Mat}_n(\Delta_{\bar{0}})$. It is clear that id is an unit in $\text{Mat}_n(\Delta_{\bar{0}})$ and $\text{id}u$ is an unit in $\text{Mat}_n(\Delta_{\bar{0}})u$.

For $n, m \in \mathbb{N}$, the set $\widetilde{\text{Mat}}_{n,m}(D)$ of $(n+m) \times (n+m)$ matrices over a division algebra D , with the chess board \mathbb{Z}_2 -grading:

$$\begin{aligned} \widetilde{\text{Mat}}_{n,m}(D)_{\bar{0}} &:= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in \text{Mat}_n(D), b \in \text{Mat}_m(D) \right\} \\ \widetilde{\text{Mat}}_{n,m}(D)_{\bar{1}} &:= \left\{ \begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix} : v \in \text{Mat}_{n \times m}(D), w \in \text{Mat}_{m \times n}(D) \right\}, \end{aligned}$$

and with multiplication given by

$$\begin{pmatrix} a_1 & v_1 \\ w_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & v_2 \\ w_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + v_1 w_2 & a_1 v_2 + v_1 b_2 \\ w_1 a_2 + b_1 w_2 & -w_1 v_2 + b_1 b_2 \end{pmatrix}$$

is a quasicrossed product. Indeed, let $a \in \text{Mat}_n(D)$ and $b \in \text{Mat}_m(D)$ be two invertible matrices, then $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is an unit in $\widetilde{\text{Mat}}_{n,m}(D)_{\bar{0}}$ with

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_R^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_L^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

Also, if $v \in \text{Mat}_{n \times m}(D)$ and $w \in \text{Mat}_{m \times n}(D)$ are invertible matrices, then $\begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix}$ is an unit in $\widetilde{\text{Mat}}_{n,m}(D)_{\bar{1}}$ with

$$\begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix}_R^{-1} = - \begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix}_L^{-1} = \begin{pmatrix} 0 & -w^{-1} \\ v^{-1} & 0 \end{pmatrix}$$

Actually, any unital antiassociative quasialgebra with semisimple even part, except the trivial one, is a quasicrossed product, because it is a finite direct sum of ideals of the form $\text{Mat}_{n_i}(\Delta^i)$ for some number n_i and some division antiassociative quasialgebra Δ^i , and ideals of the form $\widetilde{\text{Mat}}_{n_j, m_j}(D^j)$ for some division algebra D^j and some numbers n_j and m_j (see [3, Theorem 10]). As naturally expected, these examples show that there are quasicrossed products which are not division graded quasialgebras.

Example 4.5. Consider the \mathbb{Z}_n -graded quasialgebra of the deformed matrices $M_{n,\phi}(\mathbb{K})$ of the usual $n \times n$ matrices $M_n(\mathbb{K})$ with the basis elements E_{ij} of degree $j - i$, for $i, j \in \mathbb{Z}_n$,

and the multiplication

$$(X \cdot Y)_{ij} = \sum_{k=1}^n \frac{\phi(i, -k, k-j)}{\phi(-k, k, -j)} X_{ik} Y_{kj},$$

for any $X = (X_{ij})$ and $Y = (Y_{ij})$ in $M_n(\mathbb{K})$ (cf. in [8]). This \mathbb{Z}_n -graded quasialgebra is a quasicrossed product. Indeed, we easily find an invertible element in each homogeneous component of $M_{n,\phi}(\mathbb{K})$.

Remark 4.6. Observe that not all graded quasialgebras are quasicrossed products. For example, we can easily extract subalgebras of the algebra of Example 4.5 which are not quasicrossed products. The subalgebra $T_{n,\phi}(\mathbb{K})$ of the \mathbb{Z}_n -graded quasialgebra $M_{n,\phi}(\mathbb{K})$ formed by the upper triangular matrices is not a quasicrossed product. For example, the 1-dimensional homogeneous component $(T_{n,\phi}(\mathbb{K}))_{n-1}$ with basis $\{E_{1n}\}$ does not contain an invertible element.

Definition 4.7. Assume that B is an associative algebra. Given maps

$$\sigma : G \rightarrow \text{Aut}(B) \quad \text{automorphism system}$$

$$\alpha : G \times G \rightarrow U(B) \quad \text{quasicrossed mapping}$$

and a cocycle $\phi : G \times G \times G \rightarrow \mathbb{K}^\times$, we say that $(G, B, \phi, \sigma, \alpha)$ is a *quasicrossed system* for G over B if the following properties hold:

$$\sigma(g)(\sigma(h)(x)) = \alpha(g, h)\sigma(gh)(x)\alpha(g, h)^{-1} \quad (4.4)$$

$$\alpha(g, h)\alpha(gh, k) = \phi(g, h, k)\sigma(g)(\alpha(h, k))\alpha(g, hk) \quad (4.5)$$

$$\alpha(g, e) = \alpha(e, g) = 1 \quad (4.6)$$

for any $g, h, k \in G$ and $x \in B$.

Let A be a graded quasialgebra which is a quasicrossed product of G over A_e . Then for any $g \in G$ exists an unit $\bar{g} \in U(A) \cap A_g$ with $\bar{e} = 1$, and define a map $\sigma(g) : A_e \rightarrow A_e$ by

$$\sigma(g)(x) := \bar{g}x\bar{g}_R^{-1} \quad \text{for any } x \in A_e. \quad (4.7)$$

Lemma 4.8. For any $g \in G$, $\sigma(g)$ is an automorphism of A_e , meaning that for $x, y \in A_e$

$$\sigma(g)(xy) = \sigma(g)(x)\sigma(g)(y).$$

Proof. For any $g \in G$, as \bar{g} is a unit it is obvious that the map $\sigma(g)$ is bijective. Applying Lemma 3.4(i), we obtain for any $g \in G$ and $x, y \in A_e$

$$\begin{aligned} \sigma(g)(xy) &= \bar{g}xy\bar{g}_R^{-1} = \left(\bar{g}x(\bar{g}_L^{-1}\bar{g})\right)y\bar{g}_R^{-1} = \frac{1}{\phi(g, g^{-1}, g)}\left((\bar{g}x\bar{g}_L^{-1})\bar{g}\right)y\bar{g}_R^{-1} \\ &= \frac{1}{\phi(g, g^{-1}, g)}(\bar{g}x\bar{g}_L^{-1})(\bar{g}y\bar{g}_R^{-1}) = (\bar{g}x\bar{g}_R^{-1})(\bar{g}y\bar{g}_R^{-1}) = \sigma(g)(x)\sigma(g)(y) \end{aligned}$$

as desired. \square

Proposition 4.9. Let A be a graded quasialgebra which is a quasicrossed product of G over A_e . For any $g \in G$, fix a unit \bar{g} in A_g with $\bar{e} = 1$. Let $\sigma : G \rightarrow \text{Aut}(A_e)$ be the corresponding automorphism system given by Equation (4.7) and $\alpha : G \times G \rightarrow U(A_e)$ be defined by

$$\alpha(g, h) = (\bar{g}\bar{h})(\bar{g}\bar{h})_R^{-1} = \phi((gh)^{-1}, gh, (gh)^{-1})(\bar{g}\bar{h})(\bar{g}\bar{h})_L^{-1}, \quad (4.8)$$

for any $g, h \in G$. Then the following properties hold:

- (i) A is a strongly graded quasialgebra with $A_g = A_e\bar{g} = \bar{g}A_e$.
- (ii) $(G, A_e, \phi, \sigma, \alpha)$ is a quasicrossed system for G over A_e (to which we refer as corresponding to A).

- (iii) A is a free (left or right) A_e -module freely generated by the elements \bar{g} , where $g \in G$.
- (iv) For all $g, h \in G$ and $x, y \in A_e$,

$$(x\bar{g})(y\bar{h}) = x\sigma(g)(y)\alpha(g, h)\bar{gh}. \quad (4.9)$$

Conversely, for any associative algebra B and any quasicrossed system $(G, B, \phi, \sigma, \alpha)$ for G over B , the free B -module C freely generated by the elements \bar{g} , for $g \in G$, with multiplication given by Equation (4.9) (with $x, y \in B$) is a G -graded quasialgebra (with $C_g = B\bar{g}$ for all $g \in G$) which is a quasicrossed product of G over $C_e = B$ and having $(G, B, \phi, \sigma, \alpha)$ as a corresponding quasicrossed system.

Remark 4.10. We note that Proposition 4.9 generalizes the results on quasiassociative division algebras presented by H. Albuquerque and A. Santana (see Theorem 1.1 in [7] and Theorem 3.2 in [8]). The quasiassociative division algebras are precisely the quasicrossed products over the division associative algebras. Moreover, the three identities defining the multiplication in quasiassociative division algebras are now condensed in equation (4.9).

Proof. (i) Let $g \in G$ and take $u \in U(A) \cap A_g$. By Lemma 3.4(i),

$$u_L^{-1}, u_R^{-1} \in A_{g^{-1}}$$

and therefore $1 = u_L^{-1}u \in A_{g^{-1}}A_g$ and $1 = uu_R^{-1} \in A_gA_{g^{-1}}$. By Lemma 2.8, we conclude that A is a strongly graded quasialgebra. Applying Lemma 3.9(iii), $A_g = A_e\bar{g}$ and the argument of this lemma applied to left multiplication shows that $A_g = \bar{g}A_e$, proving this item.

(ii) First we prove condition (4.4). Let $g, h \in G$ and $x \in A_e$. We have

$$\begin{aligned} \sigma(g)(\sigma(h)(x)) &= \sigma(g)(\bar{h}x\bar{h}_R^{-1}) = (\bar{g}(\bar{h}(x\bar{h}_R^{-1})))\bar{g}_R^{-1} = \frac{1}{\phi(g, h, h^{-1})}(\bar{g}\bar{h})(x\bar{h}_R^{-1})\bar{g}_R^{-1} \\ &= \frac{\phi(gh, h^{-1}, g^{-1})}{\phi(g, h, h^{-1})}(\bar{g}\bar{h})(x\bar{h}_R^{-1})\bar{g}_R^{-1} = \frac{\phi(gh, h^{-1}, g^{-1})}{\phi(g, h, h^{-1})}(\bar{g}\bar{h})x(\bar{h}_R^{-1}\bar{g}_R^{-1}) \end{aligned}$$

and we get

$$\sigma(g)(\sigma(h)(x)) = \frac{\phi(gh, h^{-1}, g^{-1})}{\phi(g, h, h^{-1})}(\bar{g}\bar{h})x(\bar{h}_R^{-1}\bar{g}_R^{-1}). \quad (4.10)$$

Now, using Lemma 3.4 we observe that

$$\begin{aligned} (\bar{gh})_R^{-1}(\alpha(g, h))_R^{-1} &= (\bar{gh})_R^{-1}\left(\phi((gh)^{-1}, gh, (gh)^{-1})(\bar{g}\bar{h})(\bar{gh})_L^{-1}\right)_R^{-1} \\ &= \frac{\phi((gh)^{-1}, gh, (gh)^{-1})}{\phi((gh)^{-1}, gh, (gh)^{-1})\phi(gh, (gh)^{-1}, gh(gh)^{-1})}(\bar{gh})_R^{-1}\left((\bar{gh})_L^{-1}\right)_R^{-1}(\bar{g}\bar{h})_R^{-1} \\ &= (\bar{gh})_R^{-1}\left(\bar{gh}(\bar{g}\bar{h})_R^{-1}\right) = \frac{\phi((gh)^{-1}, gh, (gh)^{-1})}{\phi((gh)^{-1}, gh, (gh)^{-1})}(\bar{gh})_L^{-1}\bar{gh}(\bar{g}\bar{h})_R^{-1} \\ &= (\bar{gh})_L^{-1}\bar{gh}(\bar{g}\bar{h})_R^{-1} = \frac{\phi(h, h^{-1}, g^{-1})}{\phi(g, h, h^{-1}g^{-1})}\bar{h}_R^{-1}\bar{g}_R^{-1} \end{aligned}$$

and hence

$$\bar{h}_R^{-1}\bar{g}_R^{-1} = \frac{\phi(g, h, h^{-1}g^{-1})}{\phi(h, h^{-1}, g^{-1})}(\bar{gh})_R^{-1}(\alpha(g, h))_R^{-1}. \quad (4.11)$$

Applying Lemma 2.3(iii) we obtain

$$\begin{aligned} \alpha(g, h)\bar{gh} &= \left(\phi((gh)^{-1}, gh, (gh)^{-1})(\bar{g}\bar{h})(\bar{gh})_L^{-1}\right)\bar{gh} \\ &= \phi((gh)^{-1}, gh, (gh)^{-1})\phi(gh, (gh)^{-1}, gh)(\bar{g}\bar{h})\left((\bar{gh})_L^{-1}\bar{gh}\right) = \bar{g}\bar{h} \end{aligned}$$

then

$$\alpha(g, h)\overline{gh} = \overline{g}\overline{h}. \quad (4.12)$$

Returning to (4.10), using (4.11) and (4.12) we have

$$\sigma(g)(\sigma(h)(x)) = \frac{\phi(gh, h^{-1}, g^{-1})\phi(g, h, h^{-1}g^{-1})}{\phi(g, h, h^{-1})\phi(h, h^{-1}, g^{-1})}\alpha(g, h)\overline{gh}x(\overline{gh})_R^{-1}(\alpha(g, h))_R^{-1}$$

and by Lemma 2.3(iv) we conclude

$$\sigma(g)(\sigma(h)(x)) = \alpha(g, h)\sigma(gh)(x)(\alpha(g, h))_R^{-1}$$

proving (4.4). Let us now care about (4.5). For any $g, h, k \in G$, by Lemma 4.8 and using condition (4.12) we have

$$(\overline{g}\overline{h})\overline{k} = (\alpha(g, h)\overline{gh})\overline{k} = \alpha(g, h)(\overline{gh}\overline{k}) = \alpha(g, h)\alpha(gh, k)\overline{ghk}$$

On the other hand, we obtain

$$\overline{g}(\overline{h}\overline{k}) = \overline{g}(\alpha(h, k)\overline{hk}) = (\overline{g}\alpha(h, k))\overline{hk} \quad (4.13)$$

using Lemma 2.3(iii) we have

$$\begin{aligned} \sigma(g)(\alpha(h, k))\overline{g} &= (\overline{g}\alpha(h, k)\overline{g}_R^{-1})\overline{g} = \phi(g^{-1}, g, g^{-1})(\overline{g}\alpha(h, k)\overline{g}_L^{-1})\overline{g} \\ &= \phi(g^{-1}, g, g^{-1})\phi(g, g^{-1}, g)(\overline{g}\alpha(h, k))(\overline{g}_L^{-1}\overline{g}) = \overline{g}\alpha(h, k) \end{aligned}$$

Returning to (4.13)

$$\begin{aligned} \overline{g}(\overline{h}\overline{k}) &= (\overline{g}\alpha(h, k))\overline{hk} = (\sigma(g)(\alpha(h, k))\overline{g})\overline{hk} = \sigma(g)(\alpha(h, k))(\overline{g}\overline{hk}) \\ &= \sigma(g)(\alpha(h, k))\alpha(g, hk)\overline{ghk}. \end{aligned}$$

Since G is associative and $(\overline{g}\overline{h})\overline{k} = \phi(g, h, k)\overline{g}(\overline{h}\overline{k})$, we conclude that

$$\alpha(g, h)\alpha(gh, k) = \phi(g, h, k)\sigma(g)(\alpha(h, k))\alpha(g, hk)$$

proving (4.5). Because $\overline{e} = 1$ we have

$$\begin{aligned} \alpha(g, e) &= \phi((ge)^{-1}, ge, (ge)^{-1})(\overline{g}\overline{e})(\overline{ge})_L^{-1} = \phi(g^{-1}, g, g^{-1})\overline{g}\overline{g}_L^{-1} \\ &= \frac{\phi(g^{-1}, g, g^{-1})}{\phi(g^{-1}, g, g^{-1})}\overline{g}\overline{g}_R^{-1} = 1 \end{aligned}$$

thus (4.6) is also true, proving (ii).

(iii) It is a direct consequence of (i).

(iv) Let $g, h \in G$ and $x, y \in A_e$. Using Lemma 2.3(ii) we obtain

$$\begin{aligned} (x\overline{g})(y\overline{h}) &= (x\overline{g})\left((y(\overline{g}_L^{-1}\overline{g}))\overline{h}\right) = (x\overline{g})\left((y\overline{g}_L^{-1})\overline{g}\right)\overline{h} \\ &= \phi(g^{-1}, g, h)(x\overline{g})\left((y\overline{g}_L^{-1})(\overline{g}\overline{h})\right) = \frac{\phi(g^{-1}, g, h)}{\phi(g, g^{-1}, gh)}\left((x\overline{g})(y\overline{g}_L^{-1})\right)(\overline{g}\overline{h}) \\ &= \frac{\phi(g^{-1}, g, h)}{\phi(g, g^{-1}, gh)\phi(g^{-1}, g, g^{-1})}\left((x\overline{g})(y\overline{g}_R^{-1})\right)(\overline{g}\overline{h}) \\ &= x\sigma(g)(y)(\overline{g}\overline{h}) = x\sigma(g)(y)\alpha(g, h)\overline{gh} \end{aligned}$$

proving (4.9). To prove the converse, we need only to verify that the multiplication given by (4.9) is quasiassociative. To this end, let $g, h, k \in G$ and $x, y, z \in B$. As $\sigma(g) \in \text{Aut}(B)$

we have

$$\begin{aligned}
(x\bar{g})\left((y\bar{h})(z\bar{k})\right) &= (x\bar{g})\left(y\sigma(h)(z)\alpha(h,k)\overline{hk}\right) \\
&= x\sigma(g)\left(y\sigma(h)(z)\alpha(h,k)\right)\alpha(g,hk)\overline{ghk} \\
&= x\sigma(g)(y)\sigma(g)(\sigma(h)(z))\sigma(g)(\alpha(h,k))\alpha(g,hk)\overline{ghk} \\
&\stackrel{(4.4)}{=} x\sigma(g)(y)\alpha(g,h)\sigma(gh)(z)\alpha(g,h)^{-1}\sigma(g)(\alpha(h,k))\alpha(g,hk)\overline{ghk} \\
&\stackrel{(4.5)}{=} \frac{1}{\phi(g,h,k)}x\sigma(g)(y)\alpha(g,h)\sigma(gh)(z)\alpha(g,h)^{-1}\alpha(g,h)\alpha(gh,k)\overline{ghk} \\
&= \frac{1}{\phi(g,h,k)}x\sigma(g)(y)\alpha(g,h)\sigma(gh)(z)\alpha(gh,k)\overline{ghk}.
\end{aligned}$$

On the other hand,

$$\left((x\bar{g})(y\bar{h})\right)(z\bar{k}) = \left(x\sigma(g)(y)\alpha(g,h)\overline{gh}\right)(z\bar{k}) = x\sigma(g)(y)\alpha(g,h)\sigma(gh)(z)\alpha(gh,k)\overline{ghk}$$

therefore,

$$\left((x\bar{g})(y\bar{h})\right)(z\bar{k}) = \phi(g,h,k)(x\bar{g})\left((y\bar{h})(z\bar{k})\right)$$

completing the proof. \square

5. EQUIVALENCE OF QUASICROSSED PRODUCTS AND QUASICROSSED SYSTEMS

In this section we establish a relation of equivalence between quasicrossed systems, a relation between quasicrossed products and, in the end, we relate the two notions.

Definition 5.1. We say that two quasicrossed systems $(G, B, \phi, \sigma, \alpha)$ and $(G, B, \phi, \sigma', \alpha')$ over an associative algebra B for a fixed cocycle $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$ are *equivalent* if there exists a map $u : G \rightarrow U(B)$ with $u(e) = 1$ such that

$$\sigma'(g) = i_{u(g)} \circ \sigma(g) \tag{5.14}$$

$$\alpha'(g,h) = u(g)\sigma(g)(u(h))\alpha(g,h)u(gh)^{-1}, \tag{5.15}$$

for any $g, h \in G$, where $i_y(x) = yxy^{-1}$ for $x \in B$ and $y \in U(B)$.

We define an equivalence relation in the class of quasicrossed systems over an associative algebra B for a fixed cocycle $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$. Assume that a graded quasialgebra A is a quasicrossed product of G over A_e . Due to Proposition 4.9, any choice of a unit \bar{g} of A in A_g , for any $g \in G$, with $\bar{e} = 1$, determines a corresponding quasicrossed system $(G, A_e, \phi, \sigma, \alpha)$ for G over A_e , with σ and α given by

$$\begin{aligned}
\sigma(g)(x) &= \bar{g}x\bar{g}_R^{-1} \\
\alpha(g,h) &= (\bar{g}\bar{h})(\overline{gh})_R^{-1},
\end{aligned}$$

for any $x \in A_e$ and $g, h \in G$. Now, let $\{\tilde{g} : g \in G\}$ be another such set of units and $(G, A_e, \phi, \sigma', \alpha')$ be the corresponding quasicrossed system. Because $\tilde{g} \in A_g$, we infer from Proposition 4.9(i) that there is a function $u : G \rightarrow U(A_e)$ with $u(e) = 1$ such that

$$\tilde{g} = u(g)\bar{g} \quad \text{for all } g \in G.$$

We note that $u(g)$ is indeed an unit of A_e with inverse $u(g)^{-1} = \bar{g}\tilde{g}_R^{-1}$.

Lemma 5.2. *In the previous conditions we have that the quasicrossed systems $(G, A_e, \phi, \sigma, \alpha)$ and $(G, A_e, \phi, \sigma', \alpha')$ are equivalent over the associative algebra A_e .*

Proof. For $g \in G$ and $x \in A_e$ we have

$$\begin{aligned}
\sigma'(g)(x) &= \tilde{g}x\tilde{g}_R^{-1} = u(g)\bar{g}x(u(g)\bar{g})_R^{-1} = u(g)\bar{g}x\bar{g}_R^{-1}u(g)^{-1} = u(g)(\bar{g}x\bar{g}_R^{-1})u(g)^{-1} \\
&= u(g)\sigma(g)(x)u(g)^{-1} = i_{u(g)}(\sigma(g)(x))
\end{aligned}$$

proving (5.14). Now we take care of (5.15). For $g, h \in G$, using Lemma 2.3(ii)-(iii) we have

$$\begin{aligned}
u(gh)\overline{gh} &= \widetilde{gh} = \alpha'(g, h)^{-1} \widetilde{g} \widetilde{h} = \alpha'(g, h)^{-1} u(g) \overline{g} u(h) \overline{h} = \alpha'(g, h)^{-1} (u(g) \overline{g}) (u(h) (\overline{g}_L^{-1} \overline{g}) \overline{h}) \\
&= \frac{1}{\phi(g^{-1}, g, g^{-1})} \alpha'(g, h)^{-1} (u(g) \overline{g}) (u(h) (\overline{g}_R^{-1} \overline{g}) \overline{h}) \\
&= \frac{\phi(g^{-1}, g, h)}{\phi(g^{-1}, g, g^{-1})} \alpha'(g, h)^{-1} (u(g) \overline{g}) (u(h) \overline{g}_R^{-1} (\overline{g} \overline{h})) \\
&= \frac{\phi(g^{-1}, g, h)}{\phi(g^{-1}, g, g^{-1}) \phi(g, g^{-1}, gh)} \alpha'(g, h)^{-1} u(g) (\overline{g} u(h) \overline{g}_R^{-1}) (\overline{g} \overline{h}) \\
&= \alpha'(g, h)^{-1} u(g) \sigma(g) (u(h)) (\overline{g} \overline{h}) \\
&= \alpha'(g, h)^{-1} u(g) \sigma(g) (u(h)) \alpha(g, h) \overline{gh}
\end{aligned}$$

therefore

$$\alpha'(g, h) = u(g) \sigma(g) (u(h)) \alpha(g, h) u(gh)^{-1}$$

proving (5.15). Consequently, $(G, A_e, \phi, \sigma, \alpha)$ and $(G, A_e, \phi, \sigma', \alpha')$ are equivalent as desired. \square

Thus any given graded quasialgebra A which is a quasicrossed product of G over A_e determines a unique equivalence class of corresponding quasicrossed systems for G over A_e . We stress the independence of the choice of the sets of units used to define the quasicrossed systems.

Definition 5.3. Assume that A, A' are two quasicrossed products of G over A_e . We say that A and A' are *equivalent* if there is a graded isomorphism of algebras $f : A \rightarrow A'$ which is also an isomorphism of A_e -modules. The latter means that f is a isomorphism such that $f(A_g) = A'_g$ for all $g \in G$ and $f(x) = x$ for any $x \in A_e$.

Theorem 5.4. *Two quasicrossed products of G over A_e are equivalent if and only if they determine the same equivalence class of quasicrossed systems for G over A_e .*

Proof. Consider A and A' two quasicrossed products of G over A_e . Let $(G, A_e, \phi, \sigma, \alpha)$ and $(G, A_e, \phi, \sigma', \alpha')$ be the representatives of the corresponding equivalence classes of quasicrossed systems for G over A_e and take the systems of units $\{\overline{g} : g \in G\}$ and $\{\widetilde{g} : g \in G\}$ in A and A' , respectively, which give rise to the above quasicrossed systems.

First assume that A' and A are equivalent via $f : A' \rightarrow A$. Because $f(\widetilde{g}) \in A_g$ for all $g \in G$, there is a map $u : G \rightarrow U(A_e)$ with $u(e) = 1$ such that $f(\widetilde{g}) = u(g) \overline{g}$ for any $g \in G$. We observe that for given $g \in G$,

$$1 = f(1) = f(\widetilde{g} \widetilde{g}_R^{-1}) = f(\widetilde{g}) f(\widetilde{g}_R^{-1}) = u(g) \overline{g} f(\widetilde{g}_R^{-1}),$$

so $u(g)$ is an unit in A_e with inverse

$$u(g)^{-1} = \overline{g} f(\widetilde{g}_R^{-1}).$$

Now we recall the multiplication in A and A' , respectively, to be used in what follows:

$$\begin{aligned}
(x\overline{g})(y\overline{h}) &= x\sigma(g)(y)\alpha(g, h)\overline{gh} \\
(x\widetilde{g})(y\widetilde{h}) &= x\sigma'(g)(y)\alpha'(g, h)\widetilde{gh}
\end{aligned}$$

for any $x, y \in A_e$ and $g, h \in G$. To given $x \in A_e$ and $g \in G$ we have $\widetilde{g}(x\widetilde{e}) = \sigma'(g)(x)\alpha'(g, e)\widetilde{ge} = \sigma'(g)(x)\widetilde{g}$. Since f is a morphism of algebras we have

$$\begin{aligned}
f(\widetilde{g}(x\widetilde{e})) &= f(\widetilde{g}) f(x\widetilde{e}) = f(\widetilde{g}) (x f(\widetilde{e})) = (u(g) \overline{g}) (x u(e) \overline{e}) = (u(g) \overline{g}) (x \overline{e}) \\
&= u(g) \sigma(g)(x) \alpha(g, e) \overline{ge} = u(g) \sigma(g)(x) \overline{g}
\end{aligned}$$

and

$$f(\sigma'(g)(x)\tilde{g}) = \sigma'(g)(x)f(\tilde{g}) = \sigma'(g)(x)u(g)\bar{g}$$

therefore $\sigma'(g)(x) = u(g)\sigma(g)(x)u(g)^{-1}$ proving (5.14). Now for $g, h \in G$ we have $\tilde{g}h = \sigma'(g)(1)\alpha'(g, h)\tilde{g}h = \alpha'(g, h)gh$. Again, since f is a morphism of algebras,

$$f(\tilde{g}h) = f(\tilde{g})f(\tilde{h}) = (u(g)\bar{g})(u(h)\bar{h}) = u(g)\sigma(g)(u(h))\alpha(g, h)\bar{g}\bar{h}$$

and

$$f(\alpha'(g, h)\tilde{g}h) = \alpha'(g, h)f(\tilde{g}h) = \alpha'(g, h)u(gh)\bar{g}\bar{h}$$

therefore $\alpha'(g, h) = u(g)\sigma(g)(u(h))\alpha(g, h)u(gh)^{-1}$ getting (5.15). Thus (G, A_e, σ, α) and $(G, A_e, \sigma', \alpha')$ are equivalent.

Conversely, suppose that there is a map $u : G \rightarrow U(A_e)$ with $u(e) = 1$ such that (5.14) and (5.15) are verified. Using again the multiplication in A and in A' , it is easily verified that the A_e -linear extension of the map $f(\tilde{g}) = u(g)\bar{g}$ for any $g \in G$, also denoted by f , provides an equivalence of A' and A . In fact, f is an algebra morphism, because for $g, h \in G$

$$f(\tilde{g}h) = f(\alpha'(g, h)\tilde{g}h) = \alpha'(g, h)f(\tilde{g}h) = \alpha'(g, h)u(gh)\bar{g}\bar{h}$$

and

$$f(\tilde{g})f(\tilde{h}) = (u(g)\bar{g})(u(h)\bar{h}) = u(g)\sigma(g)(u(h))\alpha(g, h)\bar{g}\bar{h}$$

are equal by (5.15). Also satisfies $f(A'_g) = A_g$. Indeed, for $x_g \in A_g$, by Proposition 4.9(i) we may write $x_g = x\bar{g}$ for a certain $x \in A_e$. Then

$$f(xu(g)^{-1}\tilde{g}) = xu(g)^{-1}f(\tilde{g}) = xu(g)^{-1}u(g)\bar{g} = x\bar{g} = x_g,$$

with $xu(g)^{-1}\tilde{g} \in A'_g$. Finally, for any $x \in A_e$ we have $f(x) = f(x\tilde{e}) = xf(\tilde{e}) = x$, completing the proof. \square

Definition 5.5. Take an automorphism system $\sigma : G \rightarrow \text{Aut}(\mathbb{K})$, where we consider the field \mathbb{K} as the associative algebra B on the natural way. A quasicrossed mapping $\delta : G \times G \rightarrow \mathbb{K}^\times$ (see Definition 4.7) is called a *coboundary* if there is a function $u : G \rightarrow \mathbb{K}^\times$ such that

$$\delta(g, h) = u(g)\sigma(g)(u(h))u(gh)^{-1},$$

for any $g, h \in G$.

Proposition 5.6. The quasicrossed systems $(G, \mathbb{K}, \phi, \sigma, \alpha)$ and $(G, \mathbb{K}, \phi, \sigma, \alpha')$ over the associative algebra \mathbb{K} for a fixed cocycle $\phi : G \times G \times G \rightarrow \mathbb{K}^\times$ and an automorphism system $\sigma : G \rightarrow \text{Aut}(B)$ are equivalent if and only if $\alpha' = \delta\alpha$ for a certain coboundary δ .

Proof. Apply Theorem 5.4 considering the field \mathbb{K} as the associative algebra B on the natural way. On this context, condition (5.14) is trivial and also from \mathbb{K} is commutative we can rewrite (5.15) as $\alpha = \delta\alpha'$ where δ is a coboundary quasicrossed mapping. \square

6. CAYLEY (CLIFFORD) QUASICROSSED PRODUCTS

Let A be a finite-dimensional (not necessarily associative) algebra with identity element 1 and an *involution* $\varsigma : A \rightarrow A$, meaning that ς is an antiautomorphism ($\varsigma(ab) = \varsigma(b)\varsigma(a)$ for all $a, b \in A$) and $\varsigma^2 = id$. Moreover, the involution ς is called *strong* if it satisfies the property $a + \varsigma(a), a.\varsigma(a) \in \mathbb{K}1$, for all $a \in A$. The Cayley-Dickson process says that we can obtain a new algebra $\bar{A} = A \oplus vA$ of twice the dimension (the elements are denoted by a, va , for $a \in A$) with multiplication defined by

$$(a + vb)(c + vd) = (ac + \epsilon d\varsigma(b)) + v(\varsigma(a)d + cb),$$

and with a new involution $\overline{\varsigma}$ given by

$$\overline{\varsigma}(a + vb) = \varsigma(a) - vb,$$

for any $a, b, c, d \in A$. The symbol v here is a notation device to label the second copy of A in \overline{A} and $vv = \epsilon 1$, for a fixed nonzero element ϵ in \mathbb{K} .

Proposition 6.1. *If A is a quasicrossed product over the group G then the algebra $\overline{A} = A \oplus vA$ resulting from the Cayley-Dickson process is a quasicrossed product over the group $\overline{G} = G \times \mathbb{Z}_2$.*

Proof. First, we note that if $A = \bigoplus_{g \in G} A_g$ is a G -graded algebra, it is easy to see that $\overline{A} = A \oplus vA$ is a \overline{G} -graded algebra. We may write the grading $\overline{A} = \bigoplus_{g \in G} A_{(g,0)} \oplus \bigoplus_{g \in G} A_{(g,1)}$, with $A_{(g,0)} = A_g$ and $A_{(g,1)} = vA_g$. Now assume that $A = \bigoplus_{g \in G} A_g$ is a quasicrossed product. For any $g \in G$ exists a unit \overline{g} in A_g , so trivially we have a unit in $A_{(g,0)}$. Moreover, $v\overline{g}$ is a unity in $A_{(g,1)}$ with $v \frac{\varsigma(\overline{g}_R^{-1})}{\epsilon}$ its left inverse and right inverse, as

$$\begin{aligned} (v \frac{\varsigma(\overline{g}_R^{-1})}{\epsilon})(v\overline{g}) &= \epsilon \overline{g} \varsigma(\frac{\varsigma(\overline{g}_R^{-1})}{\epsilon}) = \epsilon \overline{g} \frac{1}{\epsilon} \varsigma^2(\overline{g}_R^{-1}) = \overline{g} \varsigma^2(\overline{g}_R^{-1}) = \overline{g} \overline{g}_R^{-1} = 1, \\ (v\overline{g})(v \frac{\varsigma(\overline{g}_R^{-1})}{\epsilon}) &= \epsilon \frac{\varsigma(\overline{g}_R^{-1})}{\epsilon} \varsigma(\overline{g}) = \varsigma(\overline{g}_R^{-1}) \varsigma(\overline{g}) = \varsigma(\overline{g} \overline{g}_R^{-1}) = \varsigma(1) = 1 \end{aligned}$$

completing the proof. \square

In [4], it was proved that after applying the Cayley-Dickson process to a $\mathbb{K}_F G$ algebra we obtain another $\mathbb{K}_{\overline{F}} \overline{G}$ algebra related to the first one which properties are predictable.

Proposition 6.2. [4] *Let G be a finite abelian group, F a cochain on it ($\mathbb{K}_F G$ is a G -graded quasialgebra). For any $s : G \rightarrow \mathbb{K}^\times$ with $s(e) = 1$ we define $\overline{G} = G \times \mathbb{Z}_2$ and on it the cochain \overline{F} and function \overline{s} ,*

$$\begin{aligned} \overline{F}(x, y) &= F(x, y), \quad \overline{F}(x, vy) = s(x)F(x, y), \\ \overline{F}(vx, y) &= F(y, x), \quad \overline{F}(vx, vy) = \epsilon s(x)F(y, x), \\ \overline{s}(x) &= s(x), \quad \overline{s}(vx) = -1, \quad \text{for all } x, y \in G. \end{aligned}$$

Here $x \equiv (x, 0)$ and $vx \equiv (x, 1)$ denote elements of \overline{G} , where $\mathbb{Z}_2 = \{0, 1\}$ with operation $1 + 1 = 0$. If $\varsigma(x) = s(x)x$ is a strong involution, then $\mathbb{K}_{\overline{F}} \overline{G}$ is the algebra obtained from Cayley-Dickson process applied to $\mathbb{K}_F G$.

As the deformed group algebras $\mathbb{K}_F G$ are quasicrossed products, we can improve yet this outcome in the following way. Any deformed group algebra $\mathbb{K}_F G$ (see Example 4.3), with a 2-cochain F on G , has a natural structure of quasicrossed system considering the associative algebra the field \mathbb{K} itself. Indeed, consider in the Definition 4.7 of quasicrossed system, $B = \mathbb{K}$, $\sigma(g) = id$ and $\alpha(g, h) = F(g, h)$, for all $g, h \in G$. Conditions (4.4) and (4.6) are trivial. From $\phi(g, h, k) = \frac{F(g, h)F(gh, k)}{F(h, k)F(g, hk)}$ for any $g, h, k \in G$, meaning that $\mathbb{K}_F G$ is a coboundary graded quasialgebra, we obtain assertion (4.5). Reciprocal, it is easy to see that any quasicrossed system $(G, \mathbb{K}, \phi, \sigma, \alpha)$, where \mathbb{K} is a field, corresponds to a deformed group algebra.

Proposition 6.3. *Assume that the deformed group algebra $\mathbb{K}_F G$ held as a quasicrossed product, corresponds to a quasicrossed system $(G, \mathbb{K}, \phi, \sigma, \alpha)$ for G over \mathbb{K} . If the algebra $\mathbb{K}_{\overline{F}} \overline{G}$ obtained from the Cayley-Dickson process applied to $\mathbb{K}_F G$ corresponds to a quasicrossed system $(\overline{G}, \mathbb{K}, \overline{\phi}, \overline{\sigma}, \overline{\alpha})$ for \overline{G} over \mathbb{K} then*

$$\begin{aligned} \overline{\alpha}(x, y) &= \alpha(x, y), \quad \overline{\alpha}(x, vy) = s(x)\alpha(x, y), \\ \overline{\alpha}(vx, y) &= \alpha(y, x), \quad \overline{\alpha}(vx, vy) = \epsilon s(x)\alpha(y, x), \quad \text{for all } x, y \in G. \end{aligned}$$

Proof. Applying Proposition 6.2 we have successively,

$$\begin{aligned}\bar{\alpha}(x, y) &= \bar{F}(x, y) = F(x, y) = \alpha(x, y), \\ \bar{\alpha}(x, vy) &= \bar{F}(x, vy) = s(x)F(x, y) = s(x)\alpha(x, y), \\ \bar{\alpha}(vx, y) &= \bar{F}(vx, y) = F(y, x) = \alpha(y, x), \\ \bar{\alpha}(vx, vy) &= \bar{F}(vx, vy) = \epsilon s(x)F(y, x) = \epsilon s(x)\alpha(y, x), \text{ for all } x, y \in G.\end{aligned}$$

□

7. SIMPLE QUASICROSSED PRODUCTS

The aim of this section is to study simple quasicrossed products. We recall the notion of simple graded quasialgebra in general.

Definition 7.1. A graded quasialgebra A is *simple* if $A^2 \neq \{0\}$ and it has no proper graded ideals, or equivalently, if the ideal generated by each nonzero homogeneous element is the whole quasialgebra.

To study simple quasicrossed product we introduce the definition of representation of a graded quasialgebra. In the following definition of modules, $A = \bigoplus_{g \in G} A_g$ is a G -graded quasialgebra with structure given by ϕ and $V = \bigoplus_{k \in G} V_k$ is a graded vector space over the same group G . We denote by \bullet the product defined in A . First we emphasize that the quasiassociative law in A is performed by $\bullet \circ (\bullet \otimes id) = \bullet \circ (id \otimes \bullet) \circ \Phi_{A,A,A}$ and can be represented by the following commutative diagram

$$\begin{array}{ccccc} A \otimes A \otimes A & \xrightarrow{\Phi_{A,A,A}} & A \otimes A \otimes A & \xrightarrow{id \otimes \bullet} & A \otimes A \\ \bullet \otimes id \downarrow & & & & \downarrow \bullet \\ A \otimes A & \xrightarrow{\bullet} & & & A \end{array}$$

Definition 7.2. Consider a degree-preserving map $\varphi : A \otimes V \longrightarrow V$ and denote $x_g.v_k := \varphi(x_g, v_k)$. If for homogeneous elements $x_g \in A_g, x_h \in A_h, v_k \in V_k$,

$$\begin{aligned}(x_g x_h).v_k &= \phi(g, h, k)x_g.(x_h.v_k), \\ 1.v_k &= v_k,\end{aligned}$$

then V is called a *left graded module* over A (or a *left A -graded-module*).

The condition of left graded module is a natural polarization of the quasiassociativity of the product on A , as we can see by the following commutative diagram

$$\begin{array}{ccccc} A \otimes A \otimes V & \xrightarrow{\Phi_{A,A,V}} & A \otimes A \otimes V & \xrightarrow{id \otimes \varphi} & A \otimes V \\ \bullet \otimes id \downarrow & & & & \downarrow \varphi \\ A \otimes V & \xrightarrow{\varphi} & & & V \end{array}$$

Definition 7.3. Consider a degree-preserving map $\psi : V \otimes A \longrightarrow V$ and denote $v_k.x_g := \psi(v_k, x_g)$. If for homogeneous elements $x_g \in A_g, x_h \in A_h, v_k \in V_k$,

$$\begin{aligned}(v_k.x_g).x_h &= \phi(k, g, h)v_k.(x_g x_h), \\ v_k.1 &= v_k,\end{aligned}$$

then V is called a *right graded module* over A (or a *right A -graded-module*).

Similarly, the condition of right graded module is represented in the following commutative diagram

$$\begin{array}{ccccc}
 V \otimes A \otimes A & \xrightarrow{\Phi_{V,A,A}} & V \otimes A \otimes A & \xrightarrow{id \otimes \bullet} & V \otimes A \\
 \psi \otimes id \downarrow & & & & \downarrow \psi \\
 V \otimes A & \xrightarrow{\psi} & & & V
 \end{array}$$

Definition 7.4. If V is a left and right graded module of A and if for homogeneous elements $x_g \in A_g, x_h \in A_h, v_k \in V_k$,

$$(x_g \cdot v_k) \cdot x_h = \phi(g, k, h) x_g \cdot (v_k \cdot x_h),$$

then V is called a *graded bimodule* over A (or an *A -graded-bimodule*).

Moreover, the condition of graded bimodule is represented by the following commutative diagram

$$\begin{array}{ccccc}
 A \otimes V \otimes A & \xrightarrow{\Phi_{A,V,A}} & A \otimes V \otimes A & \xrightarrow{id \otimes \psi} & A \otimes V \\
 \varphi \otimes id \downarrow & & & & \downarrow \varphi \\
 V \otimes A & \xrightarrow{\psi} & & & V
 \end{array}$$

Now we present some examples of graded modules over graded quasialgebras.

Example 7.5. Consider the antiassociative quasialgebra $A := \widetilde{Mat}_{2,2}(\mathbb{K})$ of the square matrices over the field \mathbb{K} graded by the group \mathbb{Z}_2 such that $A_{\bar{0}} := \langle E_{11}, E_{22} \rangle$ and $A_{\bar{1}} := \langle E_{12}, E_{21} \rangle$ satisfying the multiplication

$$\begin{pmatrix} a_1 & v_1 \\ w_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & v_2 \\ w_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + v_1 w_2 & a_1 v_2 + v_1 b_2 \\ w_1 a_2 + b_1 w_2 & -w_1 v_2 + b_1 b_2 \end{pmatrix}.$$

Consider the vector space $M := \langle m, n \rangle$ endowed with the grading by the group \mathbb{Z}_2 with $M_{\bar{0}} := \langle m \rangle$ and $M_{\bar{1}} := \langle n \rangle$ acting on A as follows:

$$\begin{aligned}
 mE_{11} &= nE_{21} = m, \quad mE_{12} = nE_{22} = n, \\
 mE_{22} &= mE_{21} = nE_{11} = nE_{12} = 0;
 \end{aligned}$$

and on the other side,

$$\begin{aligned}
 E_{22}m &= E_{21}n = m, \quad -E_{12}m = E_{11}n = n, \\
 E_{11}m &= E_{21}m = E_{22}n = E_{12}n = 0.
 \end{aligned}$$

We check easily that M is both a right A -graded-module and a left A -graded-module, although the two structures are not compatible, that is, M is not a graded bimodule over A (just note that $(E_{21}n)E_{12} = n$ and $E_{21}(nE_{12}) = 0$).

Example 7.6. We consider a commutative G -graded quasialgebra $\mathbb{K}_F G$. Applying Proposition 4.6 in [4], we know that the quasialgebra obtained from $\mathbb{K}_F G$ by the Cayley-Dickson doubling process can be defined by the same cocycle graded by G being the degree of the element vx equal to the degree of x , for $x \in G$. Then the subspace $v\mathbb{K}_F G$ constitutes an example of a graded bimodule over $\mathbb{K}_F G$.

Take the concrete case of the simple commutative non-associative algebra $\mathbb{K}_F \mathbb{Z}_3$ with multiplication defined by the table (cf. [4, Proposition 5.3]):

	e	e_1	e_2
e	e	e_1	e_2
e_1	e_1	$-e_2$	e
e_2	e_2	e	e_1

and diagonal involution given by $s(e) = 1$ and $s(e_1) = s(e_2) = -1$. Then we can obtain a \mathbb{Z}_3 -graded quasialgebra with the same cocycle graded by \mathbb{Z}_3 , constructed by the Cayley-Dickson process from $\mathbb{K}_F\mathbb{Z}_3$, where $M := v\mathbb{K}_F\mathbb{Z}_3$ is a graded bimodule over $\mathbb{K}_F\mathbb{Z}_3$ define by $M := \langle ve, ve_1, ve_2 \rangle$ endowed with the grading by the group \mathbb{Z}_3 with $M_0 := \langle ve \rangle$, $M_1 := \langle ve_1 \rangle$ and $M_2 := \langle ve_2 \rangle$ acting on $\mathbb{K}_F\mathbb{Z}_3$ on the right and on the left as follows:

	e	e_1	e_2		e	e_1	e_2	
ve	ve	ve_1	ve_2		ve	$-ve_1$	$-ve_2$	ve
ve_1	ve_1	$-ve_2$	ve		ve_1	ve_2	$-ve$	ve_1
ve_2	ve_2	ve	ve_1		ve_2	$-ve$	$-ve_1$	ve_2

Definition 7.7. Let V be an A -graded-bimodule, a *graded submodule* $W \subset V$ is a submodule (meaning $AW \subset W$) such that $W = \bigoplus_{g \in G} (W \cap V_g)$. We said that a A -graded-bimodule V is *simple* if it contains no proper graded submodules.

Example 7.8. A *graded quasialgebra* A is an A -graded-bimodule acting on itself by the product map. Also, each one $A_g \subset A$ is an A_e -graded-bimodule and a graded submodule of A , for any $g \in G$.

Definition 7.9. Consider two A -graded-bimodules V and V' . An A -linear $f : V \rightarrow V'$ is said to be a *graded morphism of degree g* if $f(V_h) \subset V'_{hg}$, for all $h \in G$.

Now we recall the definition of radical of a graded quasialgebra.

Definition 7.10. Let A be a graded quasialgebra. The radical of A is defined by the intersection

$$\text{rad}(A) = \bigcap \{ \text{ann } M : M \text{ simple graded left } A\text{-module} \},$$

where $\text{ann } M$ is the annihilator of M in A .

The radical of a graded quasialgebra A is a proper graded ideal of A . So $\text{rad}(A) = \{0\}$ if A is simple.

Theorem 7.11. Let A be a simple quasicrossed product. Then A_e is a semisimple associative algebra.

Proof. It is similar to the proof of Theorem 4.3 in [8]. Let $J(A_e)$ denote the Jacobson radical of the associative algebra A_e . Given a simple graded A -module $M = \bigoplus_{g \in G} M_g$, each M_g is a simple A_e -module. Thus if $a_0 \in J(A_e)$ then $a_0 M_g = 0$, $\forall g \in G$. Therefore $J(A_e) \subseteq \text{rad}(A) = \{0\}$ and A_e is semisimple. \square

In case $G = \mathbb{Z}_2$, the classification of quasialgebras that have semisimple associative null part was done in [3], so we have the following result.

Theorem 7.12. Any simple quasicrossed product A of \mathbb{Z}_2 over A_0 is isomorphic to one of the following algebras:

- (i) $\text{Mat}_n(\Delta)$, for some n and some division antiassociative quasialgebra Δ ;
- (ii) $\widetilde{\text{Mat}}_{n,m}(D)$, for some natural numbers n and m and some division algebra D .

Moreover, the natural numbers n and m are uniquely determined by A and so are (up to isomorphism) the division antiassociative quasialgebra Δ and the division algebra D .

8. THE $\mathbb{K}_F G$ CASE

Definition 8.1. The *centralizer* of a subset X of a graded quasialgebra A is the subset

$$C_A(X) := \{a \in A : ax = xa, \text{ for all } x \in X\}$$

In particular, if $X = A$ then $C_A(A) = C(A)$ is the *center* of the graded quasialgebra. A graded quasialgebra A over a field \mathbb{K} is called *central* if $C(A) = \mathbb{K}$.

Definition 8.2. A finite-dimensional graded quasialgebra A over a field \mathbb{K} is *central simple* if A is simple and central. In particular, if A is central the concept of central simple agree with the usual simplicity.

Example 8.3. (1) Any simple algebra is a central simple graded quasialgebra over its center.

(2) The complex number \mathbb{C} is a central simple algebra over \mathbb{C} , but not over the real numbers \mathbb{R} (the center of \mathbb{C} is all \mathbb{C} , not just \mathbb{R}).

(3) The quaternions \mathbb{H} form a 4-dimensional central simple algebra over \mathbb{R} .

Theorem 8.4. The deformed group algebra A corresponding to a given quasicrossed system $(G, \mathbb{K}, \phi, \sigma, \alpha)$ is a central simple quasialgebra over the field \mathbb{K} .

Proof. If an element $x = \sum_{g \in G} k_g \bar{g} \in A$, with $k_g \in \mathbb{K}$, belongs to the center of A , then it commutes with all elements k of \mathbb{K} , and therefore

$$\begin{aligned} \sum_{g \in G} (kk_g) \bar{g} &= (k\bar{e}) \left(\sum_{g \in G} k_g \bar{g} \right) = \left(\sum_{g \in G} k_g \bar{g} \right) (k\bar{e}) = \sum_{g \in G} k_g \sigma(g)(k) \alpha(g, e) \bar{g} \bar{e} \\ &= \sum_{g \in G} k_g \sigma(g)(k) \bar{g} \end{aligned}$$

This means that whenever $k_g \neq 0$, then $k = \sigma(g)(k)$ for all $k \in \mathbb{K}$, and consequently $\sigma(g) = id$. Thus $C_A(\mathbb{K}) = \mathbb{K}$, and so $C(A) \subset \mathbb{K}$. However, if $k \in \mathbb{K} \cap C(A)$, then $k\bar{g} = \bar{g}k = \sigma(g)(k)\bar{g}$, for all $g \in G$, and consequently $k \in \{a \in A : a\bar{g} = \sigma(g)(a)\bar{g}, \text{ for any } g \in G\} = \mathbb{K}$. Thus $C(A) = \mathbb{K}$ and the quasialgebra A is central.

Now, let I be a graded ideal of A . Choose a non-zero element $x = \sum_{g \in G} x_g \bar{g}$ in I . Multiplying x by \bar{g} , for a suitable g , we can assume that $x_e \neq 0$. Let k be an arbitrary element of \mathbb{K} . Since A is central then $kx - xk = 0$. But

$$kx - xk = \sum_{g \in G} kx_g \bar{g} - \sum_{g \in G} x_g \bar{g}k = \sum_{g \in G} kx_g \bar{g} - \sum_{g \in G} x_g \sigma(g)(k) \bar{g} = \sum_{g \in G} (kx_g - \sigma(g)(k)x_g) \bar{g}.$$

For each $g \neq e$, we have that $\sigma(g)$ is not the identity map in \mathbb{K} and by the previous calculations $x_g = 0$. So $x = x_e \bar{e}$ is an invertible element of the ideal I , hence $I = A$. Therefore A is simple as desired. \square

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